4.4 Equations Based on Conservation of Momentum

- Momentum

$$\mathbf{M} = m \cdot u$$

$$[MLT^{-1}] = [M] \cdot [LT^{-1}]$$

m = mass [M]

u = velocity $[LT^{-1}]$

- So, **M** is a vector quantity having magnitude and direction

Conservation of Momentum

Conservation of Momentum

- The time rate-of-change of momentum of a fluid is equal to the net force applied to the element

- Mathematically:

$$\frac{dM}{dt} = \Sigma F,$$

$[MLT^{-2}] = [F]$

- Derived from Newton's laws of motion

$\sum F$ = net force acting on the element

Newton’s Laws of Motion

- The momentum of a body remains constant unless a net force acts upon the body (conservation of momentum)

- The rate of change of momentum of a body is proportional to the net force acting on the body, and is in the same direction as the net force:

$$\sum F = m \cdot a \qquad a = U / t$$

$$\sum \vec{F} = \frac{D}{Dt}(M \times \vec{U})$$

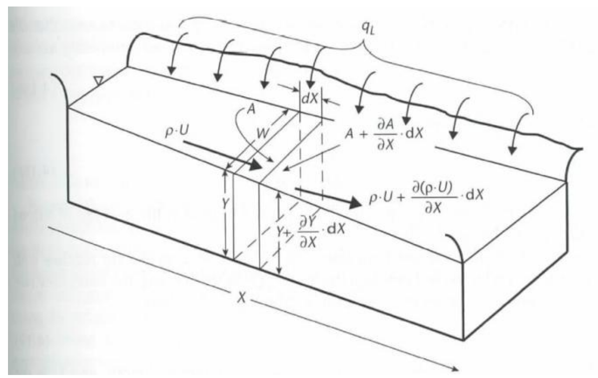
a = acceleration

M = mass

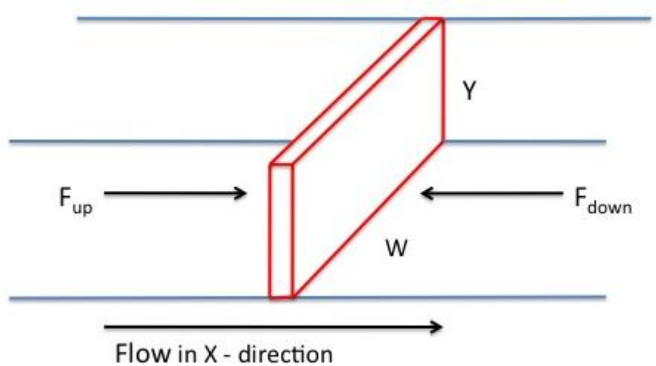
$\vec{U}$ = velocity

Deriving downstream x-direction momentum changes

- Dingman mentions that the conservation-of-momentum equation can be derived in 3-dimensional Cartesian coordinates but jumps to one-dimensional flow.
- Gives us this diagram:



- Can be simplified to:



$$\frac{dM}{dt} = \Sigma F,$$

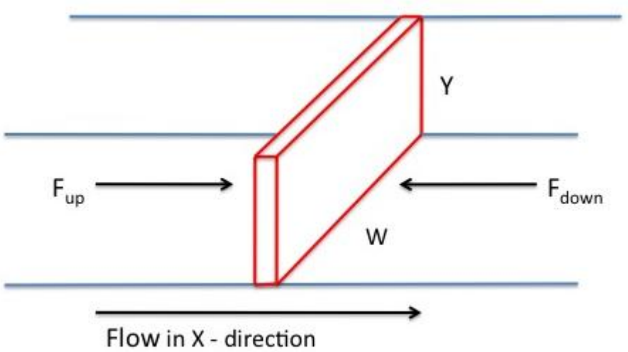
Deriving downstream X-direction momentum changes.....?

- First, let's work with the left side of the equation → $\frac{dM}{dt} = \Sigma F,$

$$\frac{dM}{dt} = \rho \cdot Q \cdot \frac{\partial U}{\partial X} \cdot dX$$

$[MLT^{-2}] = [ML^{-3}] \cdot [L^{3}T^{-1}] \cdot [LT^{-1}L^{-1}] \cdot [L]$

- Assumptions:
  - Q is spatially & temporally constant
  - ρ is constant
  - No lateral inflow



Rate of change of M for an element passing through the channel segment is due only to is DS change in U

ρ = mass density

Q = discharge

$\frac{\partial U}{\partial X}$ = change if velocity in relation to X

dX = distance in DS direction

Deriving net force

- Let's move onto the right side of the equation →

$$\frac{dM}{dt} = \boxed{\Sigma F,}$$

- Which forces are we concerned with?

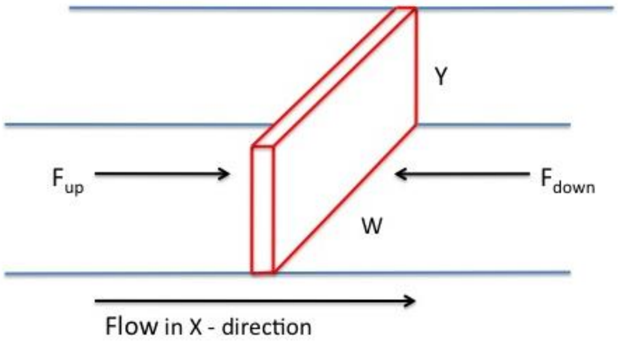
  As the DS fluid element is relatively small:

  → Ignore gravitational & frictional forces

  → Leaves us to deal with pressure forces

- Net pressure force:

$$\Sigma F = F_{up} - F_{down}$$



F_{up} = force on the US face

F_{down} = force on the DS face

Pressure force on the upstream face…..

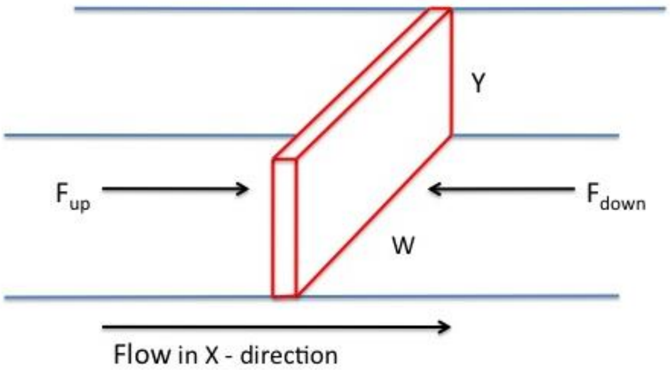
- Pressure:

$$P = \gamma \cdot Y$$

- F_{up} is the product of the **average** pressure and the area of the face:

$$F_{up} = \frac{\gamma \cdot W \cdot Y^2}{2}$$

$$= (P \cdot W \cdot Y) / 2$$



$$\frac{dM}{dt} = \Sigma F,$$

γ = weight density of water

W = width

Y = depth

Pressure force on the downstream face.....

$$F_{\text{down}} = \frac{\gamma \cdot W}{2} \cdot \left(Y + \frac{\partial Y}{\partial X} \cdot dX\right)^2 = \left(\frac{\gamma \cdot W}{2}\right) \cdot \left(Y^2 + 2 \cdot Y \cdot \frac{\partial Y}{\partial X} \cdot dX\right).$$

From here ⟶ here ???

Let's go to the board.....

$$\Sigma F = F_{\text{up}} - F_{\text{down}} = -\gamma \cdot W \cdot Y \cdot \frac{\partial Y}{\partial X} \cdot dX.$$

Pressure Width Depth Δ Depth over distance X Δ distance X

$$\frac{dM}{dt} = \Sigma F,$$

γ = weight density of water [FL^{-3}]

Y = depth [L]

W = width [L]

Putting it all back together…..

- Back to where we started:

$$\frac{dM}{dt} = \Sigma F,$$

$$\rho \cdot Q \cdot \frac{\partial U}{\partial X} = -\gamma \cdot W \cdot Y \cdot \frac{\partial Y}{\partial X}$$

- One more simplification: $\gamma = \rho \cdot g,$ $Q = W \cdot Y \cdot U,$

which leads to:

$$U \cdot \frac{\partial U}{\partial X} = -g \cdot \frac{\partial Y}{\partial X}.$$

$[LT^{-1}]$ $[LT^{-1}L^{-1}]$ = $[LT^{-2}]$ $[L^{-1}L^{1}]$

$$\frac{dM}{dt} = \Sigma F,$$

ρ = mass density

Q = discharge

$\frac{\partial U}{\partial X}$ = Δvelocity in relation to X

$\frac{\partial Y}{\partial X}$ = Δdepth in relation to X

γ = weight density of water

Y = depth

W = width

g = gravitational acceleration

One more quick note.....

$$U \cdot \frac{\partial U}{\partial X} = -g \cdot \frac{\partial Y}{\partial X}.$$

$[LT^{-1}]$ $[T^{-1}]$ = $[LT^{-2}]$

- if $\frac{\partial U}{\partial X} > 0$ (i.e. velocity increases DS) then $\frac{\partial Y}{\partial X} < 0$ (i.e. depth decreased downstream)

$$\frac{dM}{dt} = \Sigma F,$$

U = velocity

$\frac{\partial U}{\partial X}$ = Δvelocity in relation to X

g = gravitational acceleration

$\frac{\partial Y}{\partial X}$ = Δdepth in relation to X